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A One-Dimensional Model of Sedimentation Using Darcy's Law

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Abstract

Darcy's law together with the mass conservation equation is used to predict concentration profiles in a one-dimensional sedimenting column. Analytic solutions are obtained for two special cases of the diffusivity and hydraulic conductivity. For derived physical parameters the theoretical predictions are compared against experimental results from three sedimenting columns, each of different height. The predictions compare favorably with observations, indicating that this macroscopic approach of Darcy's law should be further developed, both numerically and into more than one spatial dimension.

INTRODUCTION

The separation of solid and liquid on a very large scale is one of the mining industry's major problems. Attempts to model the sedimentation process have had limited success. One of the limiting factors has been our inability to obtain accurate, independent values for the physical parameters over the entire range of observed concentrations. These physical properties determine the rates of the sedimentation process. Smiles (*6*) has experimentally determined values of the hydraulic conductivity and diffusivity at higher concentrations. However, at the dilute end of the range, where interaction effects of a colloidal suspension are still important, reliable estimates of the hydraulic conductivity and dif-

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fusivity are not available. In this paper a one-dimensional model of sedimentation that incorporates Darcy's law in the mass conservation equations is compared against results from sedimentation experiments in tall Plexiglas columns. The vertical dimension is very much greater than the horizontal one, so we are approximating one-dimensional sedimentation in the laboratory. Needless to say, this is far removed from the geometry of the shallow sedimentation ponds or lakes used in industry, but nevertheless it does give us the opportunity to observe if the physical parameters we obtain are meaningful and can lead to useful theoretical models.

In this paper the equations relevant to water infiltration in swelling porous media are applied to the essentially inverse problem of calculating concentration profiles in a (contracting) sedimenting column. In a previous paper, Blake and Colombera (*1*) compared experimentally determined concentration profiles with those from a linearized form of the equations. Here we extend the analysis to a nonlinear form of the equations, comparing the predictions with the previous linear solutions and further experimental results. These analytic solutions will prove valuable as guidelines to the full nonlinear equations once the physical parameters (hydraulic conductivity and diffusivity) are known accurately over the entire range of concentrations.

THEORY

The theory used in our analysis has been developed for water flow in swelling soils by Smiles and Rosenthal (*2*) and Philip (*3–5*), and for sedimentation by Smiles (*6*, *7*) and Blake and Colombera (*1*). The analysis is based on Darcy's law,

$$v = -K(\vartheta)\,\partial\Phi/\partial z \tag{1}$$

where v is the volume flux of water relative to the particles, K is the hydraulic conductivity which is a function of the moisture ratio ϑ (= volume of liquid/volume of solids), Φ is the potential, and z is the vertical coordinate ($z = 0$, corresponds to the base of the sedimenting column). The analysis is simplified in systems which are changing their length if we change to Lagrangian coordinates. The relation between the Lagrangian coordinate, m, and z, the physical space dimension, is

$$dm/dz = (1 + \vartheta)^{-1} = \phi \tag{2}$$

where ϕ is the conventional concentration (volume of solid/total volume).

Continuity now requires that

$$\frac{\partial \vartheta}{\partial t} = \frac{\partial}{\partial m}\left[\frac{K(\vartheta)}{1 + \vartheta}\frac{\partial \Phi}{\partial m}\right] \tag{3}$$

It is appropriate to split up the potential Φ into two parts, those due to repulsive forces between particles Ψ and those due to gravity. Hence, we define

$$\Phi = \Psi + (\gamma_c - 1)(M - m) \tag{4}$$

where γ_c is the specific gravity of the solid. We will assume, and experimental evidence (*6*) suggests, that Ψ is a function of ϑ in the case of the red mud slurries used in our experiments. For this assumption to be valid a near homogeneous particle size is required with no sorting of particles during settling. On substitution of (4) into (3), we obtain the following nonlinear Fökker-Planck equation,

$$\frac{\partial \vartheta}{\partial t} = \frac{\partial}{\partial m}\left[D_m \frac{\partial \vartheta}{\partial m}\right] - (\gamma_c - 1)\frac{dK_m}{d\vartheta}\frac{\partial \vartheta}{\partial m} \tag{5a}$$

where

$$D_m = K_m \, d\Psi/d\vartheta \tag{5b}$$

and

$$K_m = K(\vartheta)/(1 + \vartheta) \tag{5c}$$

The subscript m implies the diffusivity D_m and hydraulic conductivity K_m are taken relative to the Lagrangian coordinates. At the base of the column ($m = 0$) we apply the zero flux condition,

$$K_m \, \partial\Phi/\partial m = D_m \, \partial\vartheta/\partial m - (\gamma_c - 1)K_m = 0, \qquad m > 0, t > 0 \tag{6}$$

The boundary condition at the interface between the clear liquid and sedimenting column has not been satisfactorily resolved. Experimental physicists would argue that the condition is

$$\Psi = 0, m = M, t > 0 \tag{7}$$

However, physically this ignores the very thin Brownian motion layer that exists above the interface where there is a rapid relative change in Ψ, and mathematically it also creates problems because ϑ must then be singular at $m = M$ (i.e., $\vartheta \to \infty$). Ignoring the Brownian motion layer, the interface may be regarded both experimentally and physically to be a sharp, distinct boundary in our experiments. If any particle is displaced above the interface, it will immediately catch up as the velocity $v(\vartheta_1)$ is greater than $v(\vartheta_2)$ for $\vartheta_1 > \vartheta_2$. It is also observed that the moisture ratio ϑ at the interface remains constant at the initial value for most of the sedimentation process, at least until the effects of the zero-flux condition at the base are felt at the top. Thus both the initial ($t = 0$) and upper boundary condition ($m = M$, $t > 0$) will be

$$\vartheta = \vartheta_n \tag{8}$$

for the theoretical models in this paper.

The one-dimensional sedimentation problem is now mathematically formulated in terms of Eq. (5a), with boundary and initial conditions in Eqs. (6) and (8), provided we can experimentally or theoretically determine K_m and D_m over the whole range of ϑ. In the next section we will consider two special cases of K_m and D_m.

TWO ANALYTIC MODELS

The first model corresponds to the case when Eq. (5a) is linearized and we have the special values

$$D_m = D_0,\ K_m = k_0(\vartheta - \vartheta_0) \tag{9}$$

The second model corresponds to Eq. (5a) reducing to Burger's equation [see Hopf (*8*), Cole (*9*), Whitham (*10*)]. In this case we have the expressions

$$D_m = D_1,\ K_m = \tfrac{1}{2}k_1(\vartheta - \vartheta_1)^2 \tag{10}$$

The solution for the case defined in Eq. (9) has been obtained previously (*1*). The series solution for the moisture ratio, ϑ, physical space, z, and height of clear liquid at the top, Q, for this case is

$$\vartheta(m,t) = \vartheta_0 + (\vartheta_n - \vartheta_0)\left[e^{-2\beta(1-m')} - 4\beta e^{\beta m'}\sum_{n=1}^{\infty}\frac{\alpha_n \sin\alpha_n(1-m')e^{-(\alpha_n^2+\beta^2)t'}}{(\alpha_n^2+\beta^2)[(1+\beta)\cos\alpha_n - \alpha_n\sin\alpha_n]}\right] \tag{11a}$$

$$z(m,t) = M\left[(1+\vartheta_0)m' + (\vartheta_n - \vartheta_0)\left\{\frac{e^{-2\beta}}{2\beta}(e^{2\beta m'} - 1) - 4\beta e^{\beta m'}\sum_{n=1}^{\infty} \times \frac{\alpha_n[\alpha_n\cos\alpha_n(1-m') + \beta\sin\alpha_n(1-m')]e^{-(\alpha_n^2+\beta^2)t'}}{(\alpha_n^2+\beta^2)[(1+\beta)\cos\alpha_n - \alpha_n\sin\alpha_n]}\right\}\right] \tag{11b}$$

$$Q(t) = M(1 + \vartheta_n) - z(M,t) \tag{11c}$$

where

$$m' = m/M,\ t' = Dt/M^2,\ \beta = (\gamma_c - 1)k_0M/2D_0$$

and

$$\alpha_n \cot \alpha_n + \beta = 0; \qquad n = 1, 2, 3, \cdots \tag{11d}$$

The volume of clear liquid at the top is obtained by subtracting the present height of the column from the initial height $Z_n\ [= M(1 + \vartheta_n)]$.

In the second model (Burger's equation), the series solution is expressed as

$$\vartheta = \vartheta_1 - \frac{(\vartheta_n - \vartheta_1)\,\partial\Theta/\partial m}{\delta\Theta} \tag{12a}$$

where

$$\Theta = \frac{1 + \delta(1 - m')}{1 + \delta} - 2\delta^2 \sum_{n=1}^{\infty} \frac{\sin \alpha_n m' e^{-\alpha_n^2 t'}}{\alpha_n(\alpha_n^2 + \delta^2 + \delta)} \tag{12b}$$

and

$$z(m, t) = M\left[(1 + \vartheta_1)m' - \frac{(\vartheta_n - \vartheta_1)}{\delta} \log \Theta\,(m, t)\right] \tag{12c}$$

In this case $\delta = (\vartheta_n - \vartheta_1)k_1(\gamma_c - 1)M/2D_1$, and again α_n satisfies Eq. (11d) but with δ replacing β in the expression. For computational purposes we find Eq. (12a) does not converge very quickly for small times. We therefore use the short-time expansion

$$\Theta = \Theta_0 + \Theta_1 + \cdots \tag{13a}$$

where

$$\Theta_0 = e^{-\delta m' + \delta^2 t'} - \tfrac{1}{2} e^{\delta^2 t' - \delta m'} \operatorname{erfc}\left[\frac{m'}{2\sqrt{t'}} - \delta\sqrt{t'}\right]$$
$$-\tfrac{1}{2} e^{\delta^2 t' + \delta m'} \operatorname{erfc}\left[\frac{m'}{2\sqrt{t'}} + \delta\sqrt{t'}\right] + \operatorname{erfc}\left[\frac{m'}{2\sqrt{t'}}\right] \tag{13b}$$

and

$$\Theta_1 = -\operatorname{erfc}\left[\frac{2 - m'}{2\sqrt{t'}}\right] + 2\left(\frac{t'}{\pi}\right)^{1/2} \exp\left[-\frac{(2 - m')^2}{4t'}\right]$$
$$+ [1 - \delta(2 - m') - 2\delta^2 t'] \exp[\delta(2 - m') + \delta^2 t']$$
$$\times \operatorname{erfc}\left[\frac{2 - m'}{2\sqrt{t'}} + \delta t'\right] \tag{13c}$$

Θ_0 corresponds to the semi-infinite solution with a zero-flux boundary condition, while Θ_1 corresponds to the first reflection about the interface between the clear liquid and sediment.

We now have analytic expressions for the moisture ratio, or concentration, ϕ, where

$$\phi(m, t) = (1 + \vartheta(m', t))^{-1}$$

and real space coordinate $z(m, t)$ such that we can predict concentration profiles provided we have estimates for D_i, k_i, and ϑ_i ($i = 1, 2$). In the next section we outline a procedure for obtaining these values.

PARAMETER ESTIMATION

In Fig. 1 the dependence of K_m and D_m on ϑ are shown for previously determined experimental values (*6*) and also for the values used in the two models of this paper. The method of deriving them will be discussed later in this section. It should be emphasized that the experimentally determined values of K_m and D_m of Smiles (*6*) are only valid in the range $2 \leq \vartheta \leq 6$. Extrapolated values of the log/linear plot are shown on the diagram.

To predict the concentration profiles we need estimates of the parameters ϑ_0, k_0, and D_0 in the linear model and ϑ_1, k_1, and D_1 in the Burger model. The estimation procedure we now outline has the disadvantage that it is experiment specific in that the parameters vary from one experiment to another. On the other hand, it does enable us to predict the concentration profiles very well qualitatively and reasonably well on a quantitative basis. This is not the case for infiltration into soils which have nonlinear properties, where even qualitatively the linear models predict incorrect shapes. We estimate the parameters from (1) the initial linear rate of decrease of the interface between the clear liquid and sediment, (2) the equilibrium height of the column, and (3) the equilibrium moisture ratio at the base of the column ($m = 0$). This gives us three conditions in three unknowns. The three parameters are then obtained by using Newton's method.

If we knew the functional dependence of K_m and D_m on ϑ over the complete range of ϑ, we could easily obtain the required values for conditions (1), (2), and (3) above. However, in practice, we must often obtain these values from experimental observation on the columns. In the near future we hope to have independent methods of estimating these parameters.

Thus for short times the initial collection of clear water varies linearly with time as follows:

$$Q \sim (\gamma_c - 1)K_m(\vartheta_n)t \sim (\gamma_c - 1)k_0(\vartheta_n - \vartheta_0)t \tag{14a}$$

$$\sim \tfrac{1}{2}(\gamma_c - 1)k_1(\vartheta_n - \vartheta_1)^2 t \tag{14b}$$

Since we can obtain $K_m(\vartheta_n)$ from the experiments, we have

$$k_0(\vartheta_n - \vartheta_0) = K_m(\vartheta_n) \tag{15a}$$

and

$$\tfrac{1}{2}k_1(\vartheta_n - \vartheta_1)^2 = K_m(\vartheta_n) \tag{15b}$$

Here equations labeled with (a) refer to the linear model, (b) to the Burger model. The equilibrium volume of clear liquid at the top of the column, Q_∞, gives the following relationships:

$$Q_\infty = M(\vartheta_n - \vartheta_0)\left[1 - \frac{1}{2\beta}(1 - e^{-2\beta})\right] \tag{16a}$$

and

$$Q_\infty = M(\vartheta_n - \vartheta_1)\left[1 - \frac{\log(1 + \delta)}{\delta}\right] \tag{16b}$$

Likewise the equilibrium moisture ratio at the base, ϑ_e, yields

$$\vartheta_e = \vartheta_0 + (\vartheta_n - \vartheta_0)e^{-2\beta} \tag{17a}$$

and

$$\vartheta_e = \vartheta_1 + \frac{(\vartheta_n - \vartheta_1)}{1 + \delta} \tag{17b}$$

In the next section we compare the solutions from the two mathematical models against experiments on a red mud slurry carried out in three Plexiglas columns of different heights.

RESULTS

Experimental details relevant to this paper are outlined in Ref. *1*. Briefly, the sedimenting column of a red mud slurry from the alumina extraction process was contained within three elongated Plexiglas columns of differing dimensions. The sediment was gently stirred to obtain a constant concentration throughout the length of the column and then allowed to settle. The concentration profile as a function of height and time was obtained by the γ-ray attenuation method (*1, 11*).

In Fig. 1 the values of K_m and D_m used in the linear and Burger model are plotted against ϑ for the case of the long column. The asterisks correspond to the three values of $K_m(\vartheta_n)$ given in Table 1. It can be seen for this example that the hydraulic conductivity ($\vartheta \leq 11.3$) and diffusivity are smaller in the Burger model than in the linear model. Comparing these results to the experimental values of Smiles (*6*), it appears that the linear model is a better approximation than the Burger model. In fact, a combination of the K_m obtained here and from Smiles (*6*) could be a useful starting representation for numerical models. It should be noted that the diffusivities used in the theoretical models come from the high concentration end of the range.

Results for the three columns are shown in Figs. 2A, 2B, and 2C for different times throughout the sedimentation process. Details on parameters are given in Table 1.

As we are summing easily calculable analytical functions, the computational time is very small indeed (typically 0.3 to 0.5 sec CP time on the CSIRO CYBER 76 for each set of experiments shown in Fig. 2). However, the number of terms required in the series solutions to obtain convergence (Eqs. 11a, 11b, 12a, 12b) is dependent on the time the experiment has been running; for smaller times a larger number of terms are required, or in the case of the Burger model the short-time asymptotics (Eqs. 13a, 13b, 13c) were preferred.

Bearing in mind that we are approximating nonlinear expressions for

TABLE 1

Observed and Predicted Values of the Parameters in the Three Columns

	Column length (Z_n)		
	Short (0.35 m)	Medium (0.86 m)	Long (1.86 m)
ϑ_n	13.2	12.6	11.3
$K_m(\vartheta_n)$	2.87×10^{-7} m/sec	2.18×10^{-7} m/sec	1.84×10^{-7} m/sec
γ_c	3.1	3.1	3.1
ϑ_e	5.5	5.9	4.6
M	0.0243 m	0.0637 m	0.151 m
Q_∞	0.167 m	0.411 m	0.97 m
ϑ_0	5.4	5.9	4.6
k_0	3.7×10^{-8} m/sec	3.2×10^{-8} m/sec	2.7×10^{-8} m/sec
D_0	2.0×10^{-10} m²/sec	1.6×10^{-9} m²/sec	3.6×10^{-10} m²/sec
ϑ_1	5.1	5.8	4.5
k_1	8.8×10^{-9} m/sec	9.5×10^{-9} m/sec	8.0×10^{-9} m/sec
D_1	8.8×10^{-11} m²/sec	4.4×10^{-11} m²/sec	1.0×10^{-10} m²/sec

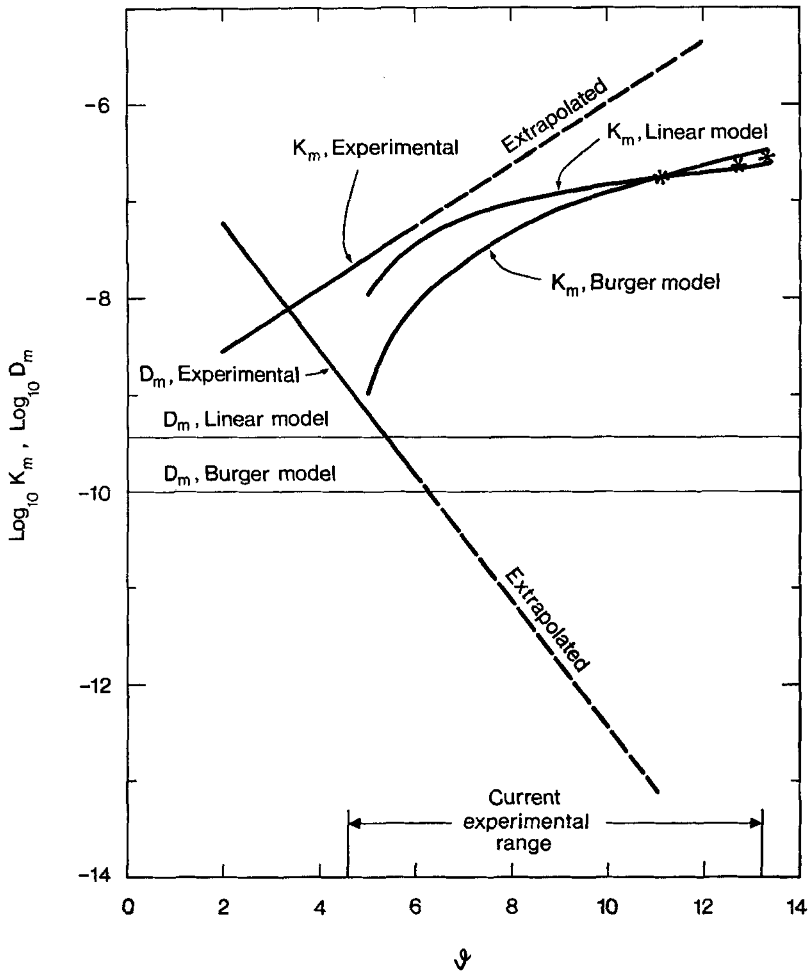

FIG. 1. Plots of the hydraulic conductivity K_m and diffusivity D_m against moisture ratio ϑ from experimentally determined values (*6*) and those derived in the two models (linear and Burger) of this paper. The asterisks (*) correspond to the values of $K_m(\vartheta_n)$ listed in Table 1. The range of values of ϑ in the experiments referred to in this paper is shown at the bottom of the diagram.

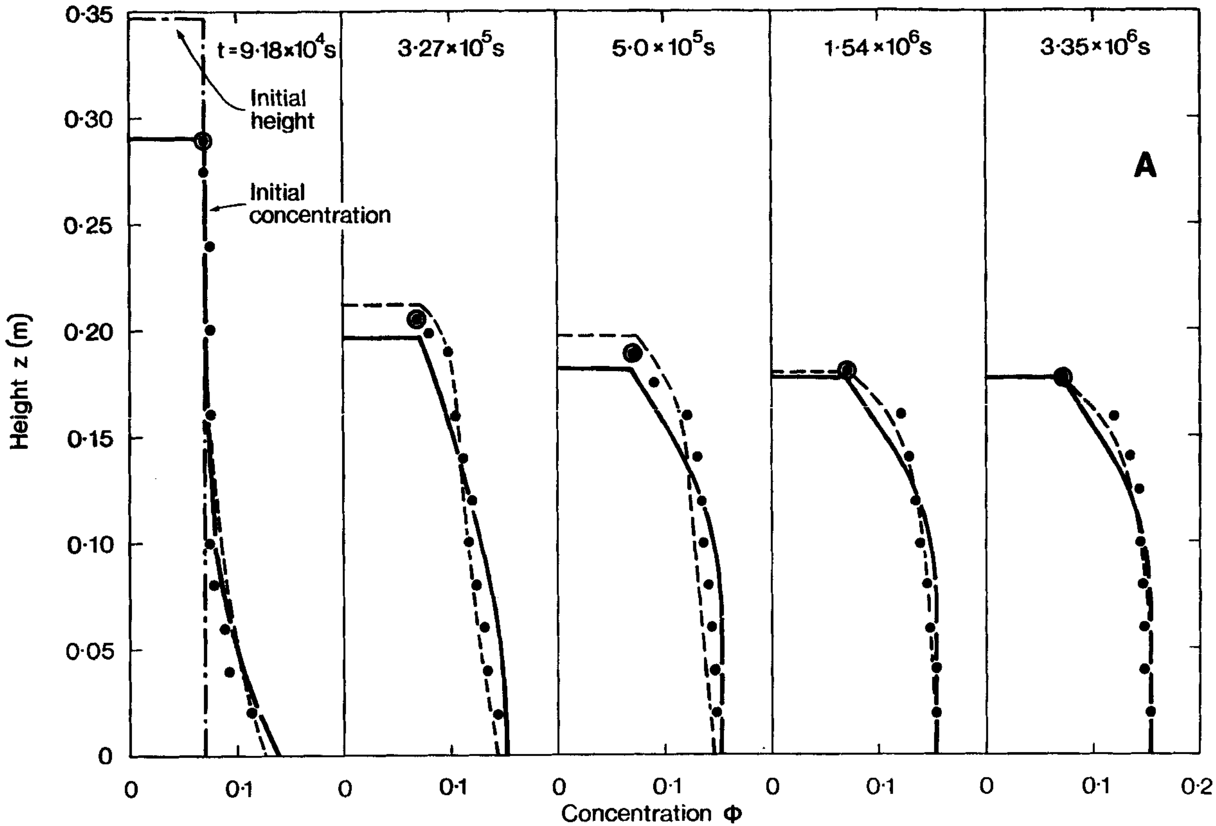

FIG. 2A. This diagram and Figs. 2B and 2C show the evolution of the concentration profiles in three sedimenting columns of initial length Z_n in (A) 0.35 m, (B) 0.86 m, and (C) 1.86 m. Details on parameters can be found in Table 1. Time in seconds is shown in the top right hand corner of each graph. The dots (●) correspond to experimental observations while ⊙ is the observed position of the interface. The solid line and broken line correspond to the linear and Burger theoretical models, respectively.

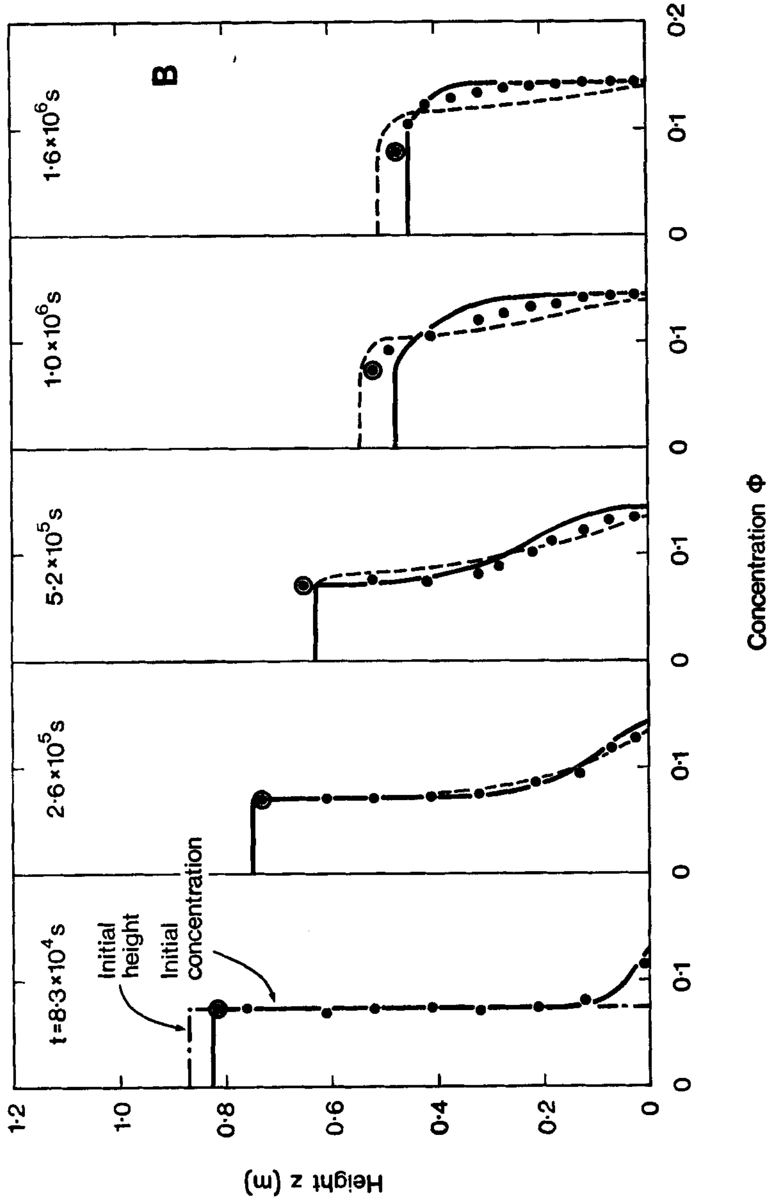

FIG. 2B. See the legend to Fig. 2A.

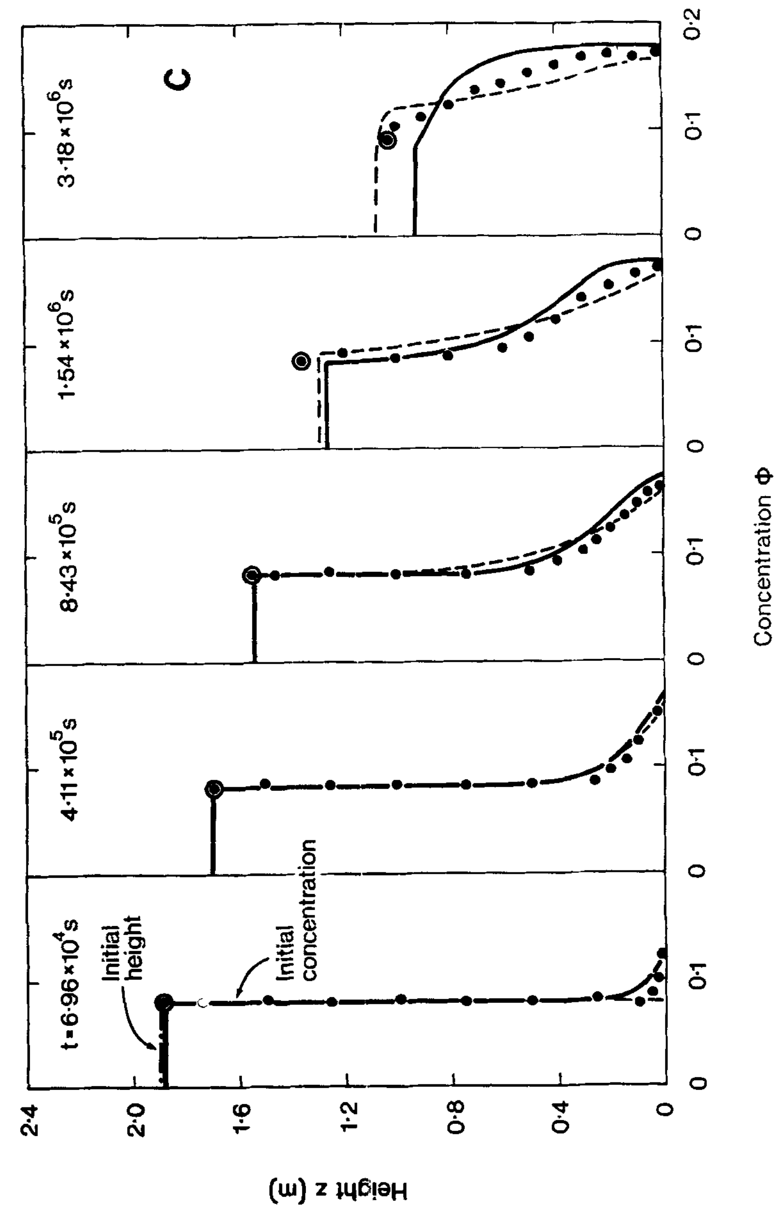

FIG. 2C. See the legend to Fig. 2A.

the diffusivity and hydraulic conductivity by an "averaged" diffusivity and a hydraulic conductivity which is either a linear or quadratic function of moisture ratio in our models, the results are reasonable. This is not so surprising because we are choosing $K_m(\vartheta_n)$ exactly and the gravitational component dominates the early parts of the sedimentation process. Furthermore, the diffusivity is derived from the balance of forces in the equilibrium profile; and provided the equilibrium concentration is relatively constant throughout the profile, an "average" diffusivity will be reasonable. Figure 1 confirms this claim as the value of the "averaged" diffusivity is at the highly concentrated end of the range (i.e., low values of ϑ).

As expected, for early times (first diagram in Fig. 2A and the first two in 2B and 2C) both models predict the profile extremely well. However, in the middle stages of the process the Burger model appears to be quantitatively more accurate, but the linear model has qualitatively the correct shape. This is due to the faster decrease in the hydraulic conductivity for the Burger model while the diffusivity is held constant, hence the increase in the concentration profile is spread out over a greater height because of the balance between the gravitational and diffusive terms in Eq. (5a).

The models presented here are part of an initial study of the mechanics of sedimenting colloidal particles. Although the models are useful, the full nonlinear equations with experimentally determined diffusivities and hydraulic conductivities will need to be solved (numerically) to ascertain the full usefulness of this approach to modeling sedimentation.

SYMBOLS

D_1	diffusivity in Burger model
D_m	diffusivity in Lagrangian coordinates
D_0	diffusivity in linear model
k_1	coefficient of conductivity in Burger model
K	hydraulic conductivity
K_m	hydraulic conductivity in Lagrangian coordinates
k_0	coefficient of conductivity in linear model
m	Lagrangian coordinate
M	total mass length of column
M'	m/M, dimensionless Lagrangian coordinate length
Q	volume of clear liquid at top
Q_∞	volume of clear liquid at top of the column at equilibrium
t	time
t'	Dt/M^2, nondimensional time

v	volume flux of water relative to particles
z	vertical spatial coordinate
Z_n	initial length of column

Greek

α_n solution of transcendental equation

$$\alpha_n \cot \alpha_n + \beta = 0$$

or

$$\alpha_n \cot \alpha_n + \delta = 0$$

β	$(\gamma_c - 1)k_0M/2D_0$, a nondimensional parameter
γ_c	specific gravity of solid component
δ	$(\vartheta_n - \vartheta_1)k_1(\gamma_c - 1)M/2D_1$, a dimensionless parameter
ϑ	moisture ratio (volume of liquid/volume of solid)
ϑ_c	equilibrium moisture ratio at base of column
ϑ_1	parameter in hydraulic conductivity in Burger model
ϑ_0	parameter in hydraulic conductivity in linear model
ϑ_n	initial moisture ratio
Θ	used in Burger equation substitution
Θ_0	first term in short time expansion of Θ
Θ_1	second term in short time expansion of Θ
ϕ	$1/(1 + \vartheta)$ concentration (volume of solid/total volume)
Φ	potential
Ψ	potential to repulsive forces between particles

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